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A Generator for the Weert Superpotential

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Weert found a superpotential $\underset{\mathrm{B}}{K}_{ijc}$ for the bounded part of the Maxwell tensor $\underset{\mathrm{B}}{T}_{ij}$ associated to the Liénard–Wiechert field. Here we obtain a fourth-rank generator $\underset{\mathrm{B}}{K}_{abcr}$ for the superpotential $\underset{\mathrm{B}}{K}_{abc}$.

In what follows we shall employ the quantities and notation given in detail by Synge (1970), López-Bonilla *et al.* (1994), Gaftoi *et al.* (1994), Aquino *et al.* (1995), and López-Bonilla *et al.* (1997a, b). The Liénard–Wiechert (LW) field is the electromagnetic field of a point charge with arbitrary motion in Minkowski space and its corresponding Maxwell tensor admits a splitting into a bounded and a radiative part (Teitelboim, 1970):

$$T_{ij} = \underset{\mathrm{B}}{T}_{ij} + \underset{\mathrm{R}}{T}_{ij} \tag{1}$$

such that

$$\underset{\mathrm{B}}{T}{}^{i}_{j,i} = 0 \tag{2}$$

$$\underset{\mathrm{R}}{T}{}^{i}_{j,i} = 0 \tag{3}$$

van Weert (1974) proved the existence of the superpotential $\underset{\mathrm{B}}{K}_{ijr} = -\underset{\mathrm{B}}{K}_{jir}$ generating the bounded part via

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$$\underset{\mathrm{B}}{T_{ij}} = \underset{\mathrm{B}}{K_i{}^r{}_{j,r}} \tag{4}$$

and thus (2) follows immediately. The bounded part is

$$\underset{\mathrm{B}}{T_{ij}} = q^2 w^{-4}\left[\frac{1}{2} g_{ij} + (k_i a_j + k_j a_i) + w^{-1}(1 - W)(k_i v_j + k_j v_i) - w^{-2}(1 - 2W)k_i k_j\right] \tag{5}$$

and (4) is satisfied by the expression

$$\underset{\mathrm{B}}{K_{bjc}} = \frac{q^2}{4} w^{-4}[w^{-1}(4W - 3)(v_b \times k_j)k_c - 4(a_b \times k_j)k_c - g_{cb}k_j + g_{cj}k_b] \tag{6}$$

It is possible to demonstrate from (6) that $\underset{\mathrm{B}}{K_{abc}}$ has the same symmetries as the Lanczos spin tensor (see, for example, Lanczos, 1962; Bampi and Caviglia, 1983; Ares de Parga, 1989; Roberts, 1989; Edgar, 1994; Gaftoi *et al.*, 1996; Edgar and Höglund, 1997):

$$\underset{\mathrm{B}}{K_{abc}} + \underset{\mathrm{B}}{K_{bca}} + \underset{\mathrm{B}}{K_{cab}} = 0, \qquad \underset{\mathrm{B}}{K^c{}_{ac}} = 0 \tag{7}$$

$$\underset{\mathrm{B}}{K_{ib}{}^r{}_{,r}} = 0 \tag{8}$$

Properties (7) and (8) are notable; they allow one to construct a "Petrov classification" for the LW field (Aquino *et al.*, 1995) which strengthens the analogies between Robinson–Trautman metrics and the LW electromagnetic field (Newman, 1974); in López-Bonilla (1997a) these symmetries (7) and (8) allow one to deduce the physical meaning of the Weert potential; it was found that $\underset{\mathrm{B}}{K_{ijr}}$ behaves like a density for the intrinsic angular momentum of the LW field.

Next we show the following original result: We explicitly prove the existence of a nonlocal generator for (6) via

$$\underset{\mathrm{B}}{K_{ibr}} = \underset{\mathrm{B}}{K_{ibr}{}^c{}_{,c}} \tag{9}$$

where (8) follows immediately from the property $\underset{\mathrm{B}}{K_{ibrc}} = -\underset{\mathrm{B}}{K_{ibcr}}$.

The first step is to notice the splitting of expression (6) into two potentials (López, 1978)

$$\underset{\mathrm{B}}{K_{ibr}} = \underset{\mathrm{B}}{\bar{K}_{ibr}} + \underset{\mathrm{B}}{\tilde{K}_{ibr}} \tag{10}$$

given by

$$\underset{\mathrm{B}}{\bar{K}_{ibr}} = \frac{q^2}{4} w^{-4} [3w^{-1}(v_b \times k_i)k_r + g_{rb}k_i - g_{ri}k_b] \tag{11}$$

$$\underset{\mathrm{B}}{\tilde{K}_{ibr}} = q^2 w^{-4} [w^{-1} W(v_i \times k_b) - (a_i \times k_b)]k_r \tag{12}$$

which also satisfy the Lanczos spin-tensor symmetries (7) and (8). It is clear from (10) that relation (9) will be proved if we construct generators for each part (11) and (12). We have no problem in the case of $\underset{\mathrm{B}}{\bar{K}_{ibr}}$ because Rowe (1978) obtained

$$\underset{\mathrm{B}}{\bar{K}_{ibr}} = \left(\frac{q^2}{4} w^{-4} D_{ibr}{}^{c}\right)_{,c} \tag{13}$$

where D_{abcr} is a tensor employed by Synge (1970) in another context and is given by

$$D_{ibr}{}^{c} = (q_{ri}k_b - g_{rb}k_i)k^c + (\delta^c_b k_i - \delta^c_i k_b)k_r \tag{14}$$

Our main contribution is the generator for (12) via the expression

$$\underset{\mathrm{B}}{\tilde{\mathbf{K}}_{ibr}} = \left[2qF^c_r p_{(\sigma)} p_{(\theta)} \int_0^\tau a_{(\theta)}(v_i \times e_{(\sigma)b})\, d\tau - q^2 w^{-4} W D_{ibr}{}^{c}\right]_{,c} \tag{15}$$

which together with (10) and (13) implies (9), the generator given by the expression

$$\underset{\mathrm{B}}{K_{ibr}{}^{c}} = 2qF^c_r p_{(\sigma)} p_{(\theta)} \int_0^\tau a_{(\theta)}(v_i \times e_{(\sigma)b})\, d\tau + \frac{q^2}{4} w^{-4}(1 - 4W) D_{ibr}{}^{c} \tag{16}$$

where we note its dependence on the worldline of the charge, so it is nonlocal; (4) and (9) imply the following expression for the bounded part of the Maxwell tensor:

$$\underset{\mathrm{B}}{T_{ij}} = \left(\underset{\mathrm{B}}{K_i{}^{r}{}_{j}{}^{c}}\right)_{,rc} \tag{17}$$

For the radiative part, (3) implies

$$\underset{\mathrm{R}}{T_{jc}} = \underset{\mathrm{R}}{K_j{}^{b}{}_{c,b}} = q^2 w^{-4}(a^2 - w^{-2} W^2) k_j k_c \tag{18}$$

with the following nonlocal potential (López-Bonilla *et al.*, 1994, 1997a, b; Aquino *et al.*, 1993; Núñez-Yépez *et al.*, 1997):

$$\underset{\mathrm{R}}{K}_{jbc} = -\frac{q^2}{4} w^{-2} [w^{-2}W^2(g_{cj}k_b - g_{cb}k_j) + w^{-1}W(v_j \times k_b)(a_c - 3w^{-2}Wk_c) + (a_j \times k_b)(4w^{-2}Wk_c - a_c)] - 2qF_{bj}p_{(\sigma)}p_{(\gamma)} \times \left[\int_0^\tau a_{(\sigma)}a_{(\gamma)}v_c \, d\tau + p_{(\beta)}\int_0^\tau a_{(\sigma)}a_{(\gamma)}e_{(\beta)c} \, d\tau\right] \tag{19}$$

but this tensor do not satisfy symmetries (7) and (8) and it is not possible to guarantee that $\underset{\mathrm{R}}{K}_{ibr}$ may be generated by an expression similar to (9).

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